Correct rounding in Double Extended Precision

Sélène Corbineau and Paul Zimmermann

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Plan

Double extended precision and correct rounding

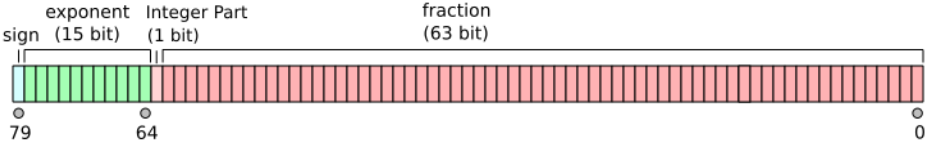
Current state of the art

Converting between double extended and double-double

An example: `expl`

Results

Double Extended Precision

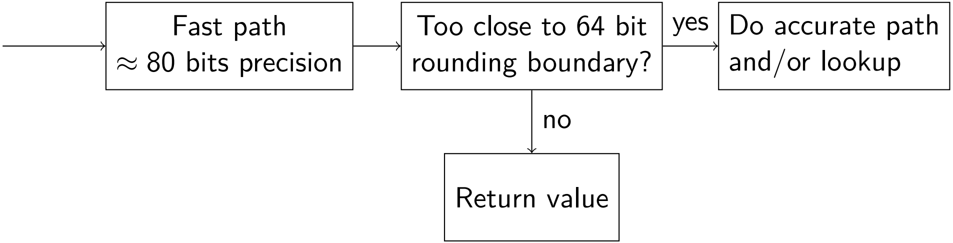


No correctly rounded routines available in double extended precision.
Existing routines are slow.

Figure by BillF4, CC BY-SA 3.0

Strategy

We follow Ziv's method to achieve correct rounding:



We implement the fast path with double precision arithmetic, avoiding x87 instructions.

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Current state-of-the-art

GNU libc, development version, with patch to benchmark some long double functions, on Intel Core i7-8700 with gcc 14.2.0. Timings in cycles.

function	min	mean	max
fmal	556	678	1030
expl	137	142	351
log2l	57	174	554
powl	700	743	1156

Plan

Double-double arithmetic

Double-double arithmetic uses pairs $(a, b) \in \mathbb{F}_{64}^2$ to represent $a + b$.

- we have product, sum primitives with explicit error bounds
- when $|b| < \mathrm{ulp}(a)$ table lookups are easy

We can expect $\approx 53 \times 2 = 106$ bits of precision, more than `long double`'s 64.

Converting from double extended precision

Input: $x \in \mathbb{F}_{80}$
Output: $a, b \in \mathbb{F}_{64}$ with $|b| < \mathrm{ulp}(a)$ and $a + b = x$

1: $a \leftarrow \circ_{64}(x)$
2: $b \leftarrow \circ_{64}(x - \circ_{80}(a))$

This only works if the exponent of x fits in the `double` exponent range.
Implemented routines deal with large/small inputs differently:

- exponentials saturate to $0, +\infty$ or 1
- logarithms keep track of x's exponent separately

Converting to double extended precision

Starting from $(a, b) \in \mathbb{F}_{64}^2$ with $|b| < \mathrm{ulp}(a)$:

1. Compute $q \approx a + b$ on 128 bits
2. Round q to `long double` and compute distance to rounding boundary

Computing q

Input: $a, b \in \mathbb{F}_{64}$ not denormals with $|b| < \mathrm{ulp}(a)$
Output: $q = (q_s, q_e, q_m) \in \mathbb{F}_{128}$ such that $(-1)^{a_s} 2^{q_e - 127} \cdot q_m \approx a + b$

1: $q_e \leftarrow a_e$
2: $q_m \leftarrow 2^{127} + 2^{64+11} a_m + (-1)^{b_s - a_s} (2^{63} + 2^{11} b_m) 2^{64 + b_e - a_e}$
3: **if** $q_m < 2^{127}$ **then**
4: $\quad q_e \leftarrow q_e - 1$
5: $\quad q_m \leftarrow 2 q_m$
return $q = (a_s, q_e, q_m)$

Assumption $|b| < \mathrm{ulp}(a)$ ensures no overflow in line 2.
There is a small error due to truncation.

Rounding q

Input: $q = (q_s, q_e, q_m)$

Output: $(y, \delta) \in \mathbb{F}_{64} \times \mathbb{Z}_{64}$ where y rounds q upwards and δ is a scaled rounding error.

write $q_m = m_h 2^{64} + m_\ell$ with $0 \le m_h, m_\ell < 2^{64}$

$\delta \leftarrow m_\ell \operatorname{cmod} 2^{64}$ ▷ $-2^{63} \le \delta < 2^{63}$

if $m_\ell \neq 0$ and $q_s > 0$ **then**

    $m_h \leftarrow m_h + 1$

    if $m_h = 2^{64}$ **then**

        $q_e \leftarrow q_e + 1, \quad m_h \leftarrow 2^{63}$

        $\delta \leftarrow \delta/2$ ▷ rounded towards zero

if $q_e \ge 16384$ **then**

    return $((-1)^{q_s}\infty, \delta)$

return $(\mathbb{F}_{64}(q_s, q_e, m_h), \delta)$

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Evaluation strategy

Inputs with exponent < -64 or ≥ 14 round trivially.
Else, let $x' = x/\log 2$ and use

$$x' = n + \frac{f}{2^{20}} + (y_h + y_\ell)$$

$$e^x = 2^{x'} = 2^n \cdot 2^{f/2^{20}} \cdot 2^{y_h + y_\ell}$$

Splitting x'

$$x' = n + \frac{f}{2^{20}} + (y_h + y_\ell)$$

1. Split x as double-double
2. Multiply by $1/\log 2$ as double-double
3. Clobber the high bits to get n, f
4. Normalize the remainder as y_h, y_ℓ with $|y_\ell| < \mathrm{ulp}(y_h)$

Evaluating the exponential

$$e^x = 2^{x'} = 2^n \cdot 2^{f/2^{20}} \cdot 2^{y_h + y_\ell}$$

1. $2^{f/2^{20}}$ is computed with a few table lookups
2. $2^{y_h + y_\ell}$ is evaluated by a degree 3 polynomial

All computations are done in double-double. We add n to the final exponent, after rounding (subnormals are treated apart).

Reconstruction

We round the previous double-double value to extended precision.
We compare $|\delta|$ to the relative computation error:

- If $|\delta| > 2^{41}$ we can guarantee correct rounding
- Otherwise, we go to the accurate path.

In practice, the test fails with probability $\approx 2^{-22}$.

Computing hard-to-round cases

We use the BaCSeL software tool (`https://gitlab.inria.fr/zimmerma/bacsel`).

- for $-2^4 < x < -2^{-65}$ and $2^{-65} \leq x <$ `0x1.484p+9`, we found 158,662 inputs with at least 54 identical bits after the round bit. Apart special cases, the largest number is 75 for $x =$ `-0x1.625ac7bfa54aba72p-14`.
- for $x \leq -2^4$ and `0x1.484p+9` $\leq x$, we search hard-to-round cases with at least 101 identical bits after the round bit. We found none.

Accurate path

Same scheme as the fast path.

Home-made 192-bit arithmetic (using 3 integer words of 64 bits).

Relative error bound is $2^{-167.006}$.

Reuse the fast path lookup tables using Markstein's "accurate table" trick.

Use the 7th degree Taylor polynomial.

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Implementation

We implemented using this scheme in CORE-MATH:

- `expl`, `exp2l`
- `log2l`
- `powl`, which was more challenging due to the dynamic ranges involved. b

Other functions (`cbrtl`, `hypotl`, `rsqrtl`) were implemented using a different scheme.

Performance

	expl	powl	log2l
CORE-MATH	47.5	165.7	44.9
Intel Math Library (2025.0.0)	64.2	288.4	83.1
GNU Libc 2.40	127.1	761.6	65.0
Openlibm 0.8.5	151.5	640.1	151.1
Musl 1.2.5	115.0	546.5	47.3

Figure: Reciprocal throughput in cycles on an Intel Xeon Silver 4214 and GCC 14.2.0

Our routines only use a few kilobytes of lookup tables.

Conclusion

- We implemented correctly rounded routines for double extended precision
- Avoiding x87 enables fast and more portable double extended precision routines
- Might be used even on processor without double extended support
- Still many functions to implement...